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TURBULENT BOUNDARY LAYER ON A FLAT PLATE IN A STREAM OF DISSOCIATING GAS

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Abstract—This paper gives a theoretical solution of a turbulent boundary layer problem of dissociating gas on a flat plate. A half-empirical Prandtl–Kármán turbulent theory is used to solve this problem.

Résumé—Cet article donne une solution théorique du problème de la couche limite turbulente sur une plaque plane dans un écoulement de gaz dissocié. Une théorie semi-empirique de Prandtl-Kármán sui la turbulence a été utilisée pour résoudre ce problème.

Zusammenfassung—Es wird eine theoretische Lösung der turbulenten Grenzschicht eines dissoziierenden Gases an der ebenen Platte mitgeteilt. Heirzu wird eine halbempirische Theorie nach Prandtl und und Kármán herangezogen.

Abstract—В статье излагается теоретическое решение задачи о турбулентном пограничном слое диссоциирующего газа на плоской пластине. Для решения этой задачи привлекается полуэмпирическая теория турбулентности Прандтля-Кармана.

THE writing of this paper developed from certain studies on turbulent boundary layer on a strongly cooled flat plate flowed by the non-gradient dissociating gas with assumed existence of thermodynamic equilibrium. The problem is determined by:

(1) Prandtl numbers of laminar sublayer and in turbulent region of layer are equal to unity.
(2) Diffusion Prandtl numbers of laminar sublayer and turbulent region of a layer are also equal to unity.
(3) The wall temperature over a plate is uniform and constant.
(4) A perfect purified two-atomic gas is being discussed.

Boundary layer equations at the above-mentioned conditions can be written in the following forms:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} + \mu_T \frac{\partial u}{\partial y}\right) - \overline{\rho' v'}\,\frac{\partial u}{\partial y} \quad (1)$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\lambda}{c_p}\frac{\partial H}{\partial y} + \frac{\lambda_T}{c_p}\frac{\partial H}{\partial y}\right) - \overline{\rho' v'}\,\frac{\partial H}{\partial y} \quad (2)$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \quad (3)$$

the equations of state for the perfect dissociating gas can be written in the form (2)

$$P = \rho R T(1 + \alpha) \quad (4)$$

where $R = gR_b$; as numbers $\Pr = 1, \Pr_T = 1$, then the equations of energy and motion will have the same form. All this allows us to consider both velocity profiles and total enthalpies as similar.

Choosing the condition of identity for the equations of energy and motion we shall have the ratio:

$$\frac{H - H_w}{H_0 - H_w} = \frac{u}{u_0} \quad (5)$$

the ratio (5) makes it possible to establish the dependence between velocities fields, densities, atom concentrations fields which are the results of dissociation and temperature fields.

Referring to Lighthill [2], we write the value of total enthalpy in the form:

$$\bar{H} = \bar{T}(4 + \alpha) + \alpha + \frac{u^2}{2RT_d} \tag{6}$$

where $\bar{H} = \dfrac{H}{RT_d}$; $\bar{T} = \dfrac{T}{T_d}$

At the presence of thermodynamic equilibrium the rate of dissociation can be determined according to formula (2):

$$\frac{\alpha^2}{1-\alpha} = \frac{1}{\bar{\rho}} \exp\left(-\frac{1}{\bar{T}}\right) \tag{7}$$

Using equations (4) to (7) we get the formulae describing the dependence between the parameters of flow at any point of the boundary layer:

$$\alpha^2 \frac{4+\alpha}{1-\alpha^2} = \frac{\bar{H}_w}{\bar{\rho}_w \bar{T}_w}\left[-\bar{R}_w^2\varphi^2 + \beta\varphi + 1 - \frac{\alpha}{\bar{H}_w}\right]$$

$$\exp\left\{\frac{4+\alpha}{\bar{H}_w(-\bar{R}_w^2\varphi + \beta\varphi + 1 - \alpha/\bar{H}_w)}\right\} \tag{8}$$

or briefly:

$$\alpha = f(\varphi) \tag{8a}$$

The form of the function $f(\varphi)$ is determined by the parameters of the external flow and by the conditions on the wall

$$\bar{R}_w^2, \beta, \bar{H}_w, \bar{T}_w$$

$$\bar{T} = \bar{H}_w\left[-\bar{R}_w^2\varphi^2 + \beta\varphi + 1 - \frac{f(\varphi)}{\bar{H}_w}\right]\frac{1}{4+f(\varphi)} \tag{9}$$

$$\frac{\rho}{\rho_w} = \bar{T}_w/(\bar{H}_w[-\bar{R}_w^2\varphi^2 + \beta\varphi + 1 - $$

$$- f(\varphi)/\bar{H}_w]\{1 + f(\varphi)\}/\{4 + f(\varphi)\}) \tag{10}$$

Here we have:

$$\beta = \frac{\bar{H}_0 - \bar{H}_w}{H_w}; \varphi = \frac{u}{u_0}; \bar{R}_w^2 = \frac{u_0^2}{2\mathrm{H}_w}$$

Velocity profile at the turbulent region of boundary layer can be defined by the approximate method.

When in equation (1) (the left-hand side of the equation) the forces of inertia of the average flow are neglected in comparison with the forces of "apparent" viscosity and using the scheme of Prandtl turbulent mixing then we shall get the following expression [1]:

$$\rho l \frac{\partial u}{\partial y} = \sqrt{\Omega} \tag{11}$$

where $\Omega = \rho_w \tau_w$† and $l = xy$

From (11) and using (10) we shall get the following formula for velocities profile:

$$\ln\eta = M\int_1^{\varphi} d\varphi/([-\bar{R}_w^2\varphi^2 + \beta\varphi + 1 - $$

$$- f(\varphi)/\bar{H}_w][1 - 3/\{4 + f(\varphi)\}]) \tag{12}$$

where

$$M = \frac{\bar{T}_w}{\bar{H}_w} x\,\xi;\ \xi = \frac{u_0}{\sqrt{(\tau_w/\rho_w)}};\ \eta = \frac{y}{\delta}$$

In every actual case the form of $f(\varphi)$ can be defined from (8), and, then it is possible to integrate the left-hand side of (12) in this way or other.

The expression $[1 - 3/\{4 + f(\varphi)\}]$ may be taking into account for the slightest changes of the rates of dissociation as a constant value equal to Q, and for $f(\varphi)$ we take the following approximating formula:

$$f(\varphi) = \phi\ 1/\rho$$

where

$$\phi = f(\bar{P})$$

In this case we shall have the following formula for the profile of velocities

$$\eta = \left[\frac{a-\varphi}{b+\varphi}\cdot\frac{1}{C}\right]^{x\xi 1/s} \tag{13}$$

† We can find the values of integration constant Ω at conversion of velocities profile, with $\rho \to \rho_{const.}$ into Prandtl logarithmic profile.

where

$$a = \sqrt{\left\{\frac{1}{\bar{R}_w^2} + \left(\frac{\beta}{2\bar{R}_w^2}\right)^2\right\}} - \frac{\beta}{2\bar{R}_w^2}$$

$$b = \sqrt{\left\{\frac{1}{\bar{R}_w^2} + \left(\frac{\beta}{2\bar{R}_w^2}\right)^2\right\}} + \frac{\beta}{2\bar{R}_w^2}$$

$$C = \frac{a+1}{b-1}$$

$$S = \sqrt{\left\{\frac{1}{\bar{R}_w^2} + \left(\frac{\beta}{2\bar{R}_w^2}\right)^2\right\}} \cdot 2\bar{R}_w^2$$

$$\xi_i = \frac{u_0}{\sqrt{(\rho_w \tau_w)}} \cdot N_w \rho_d;$$

$$N_w = \frac{\bar{\rho}_w \bar{T}_w + Q\varphi}{4\bar{T}_w Q}$$

If we neglect the forces of inertia then the equation of motion at the region of laminar sublayer is transformed to the form

$$\frac{\partial \tau}{\partial y} = 0 \qquad (14)$$

where $\tau = \mu \dfrac{\partial u}{\partial y}$

The integration (14) of $y = 0$ to y will give

$$\tau_w - \mu \frac{\partial u}{\partial y} = 0 \qquad (15)$$

The value of μ entering the equation (14) is the function of state parameters which generally speaking are changing through the thickness of the sublayer.

However, the determination of the values of μ in equation (15) we should produce in such a way that we might compensate the neglect of inertia forces from the point of establishing the velocities profile in sublayer. Let us make the most simple assumption

$$\mu = \mu_w = \text{const.}$$

In this case we have a linear profile of velocities

$$\frac{y}{\delta} = \xi^2 \frac{1}{\mathrm{Re}_{\delta_w}} \cdot \frac{u}{u_0} \qquad (16)$$

where

$$\mathrm{Re}_{\delta_w} = \frac{u_0 \delta}{\nu_w}$$

The region where we shall have a profile (16), i.e. the thickness of the laminar sublayer, we shall define by the second empirical coefficient of the turbulent theory, i.e. by Kármán number, referring the physical parameters to the wall temperature:

$$\frac{V_* \delta_0}{\nu_w} = \omega \dagger \qquad (17)$$

where $\omega = 11{\cdot}5$ and $V_* = \sqrt{(\tau_w/\rho_w)}$.

The formulae (17) and (16) give the possibility to determine the value of velocity on the external boundary of the laminar sublayer

$$\frac{u_l}{u_0} = \frac{\omega}{\xi^2} \qquad (18)$$

We get the law of resistance using the formulae (18) and (12):

$$\ln\left[\xi \cdot \frac{1}{(\mathrm{Re}_\vartheta)_w} \frac{\vartheta}{\delta}\,\omega\right] = M \int_1^{\varphi = \omega/\xi} d\varphi/([-\bar{R}_w^2\varphi^2 + \beta\varphi + 1 - f(\varphi)/\bar{H}_w] \; [1 - 3/\{4 + f(\varphi)\}]) \qquad (19)$$

where

$$\frac{\vartheta_w}{\delta} = \int_0^1 \frac{\rho}{\rho_w} \varphi(1-\varphi)\, d\eta$$

The value of ϑ_w/δ may be calculated substituting into (20) the expression for velocity profile (12) and the expression for densities profile (10).‡

Thus the formula (12) presents a relation between the coefficients of friction ξ and number $\mathrm{Re}_{\vartheta_w} = \vartheta_w u_0/\nu_w$. Taking into account this dependence it is possible to integrate the equation of motion written in the integral form and receive the distribution of friction coefficients along the plate:

$$Re_x = \int_0^{(\mathrm{Re})_{\vartheta w}} \xi^2 \, d(\mathrm{Re})_{\vartheta_w} \qquad (21)$$

† It is quite possible to make an assumption which is anologous to that we had in the reference [1]. However, it does not bring the essential accuracy.

‡ The use of velocity profile as (13) gives more simple expression for the law of resistance.

The counting of heat flows may be made on the basis of the existence of similarity of the profiles of velocities and enthalpies.

CONCLUSION

The above-mentioned method of calculation makes it possible to determine both friction and heat exchange in the flow of dissociating gas moving at high velocities.

NOMENCLATURE

x = distance along plate (m);
y = distance from plate (m);
u = gas velocity component, directed along plate (m/sec);
u' = velocity pulsation u;
v = gas velocity component, directed along axis y (m/sec);
v' = velocity pulsation v (m/sec);
ρ = gas density (kg sec^2/m^4);
ρ' = gas density pulsation (kg sec^2 m^{-4});
δ = boundary layer thickness (m),
δ_0 = laminar sublayer thickness (m);
ϑ = impulse loss thickness (m);
α = dissociation rate;
H = total enthalpy;
τ = friction stress (kg/m^2);
l = length of mixture path (m);
T = thermodynamical temperature (°K);
R_b = gas constant consisting of molecule A_2 (kg m/kg °C);
P = pressure (kg/m^2);
T_d = characteristic temperature (°K);
P_d = characteristic pressure (kg/m^2);
$\mathrm{Pr} = \frac{c_p \mu}{\lambda}$ = Prandtl criterion of laminar sublayer;
$\mathrm{Pr}_T = \frac{c_p \mu_T}{\lambda_T}$ = Prandtl criterion of boundary layer turbulent section;
g = gravity acceleration (m/sec^2);

Indices

w = flow parameters on wall;
0 = flow parameters on external boundary of layer.

REFERENCES

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